• Lecture #22 • July 9, 2015

• Qualitative Method for Systems of ODE

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$ RHS does not depend on t
Autonomous System

• With initial condition specified as:

$$\begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$$

→ Denote solution by...

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

• If $F(x,y)$, $G(x,y)$ are continuously differentiable (F_x, F_y, G_x, G_y all continuous) near (x_0, y_0), then the solution to the IVP exsist near $t = t_0$ (Exsistence & Uniqueness Theorem)

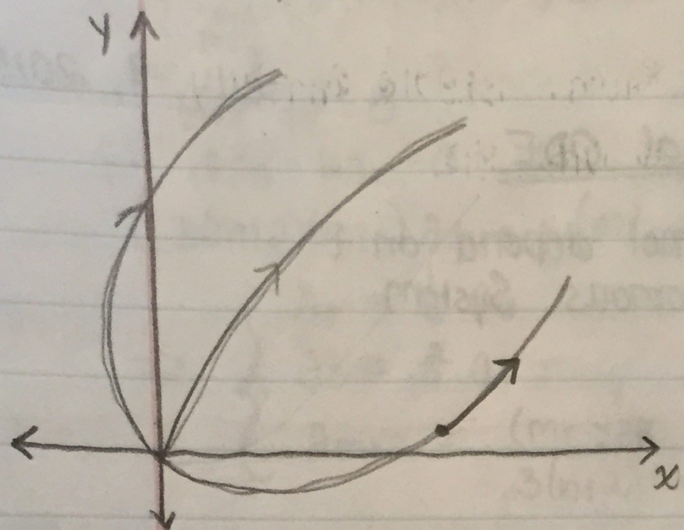
The Solution $\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$ gives a curve over the xy-plane. Although we cannot figure out $\varphi(t)$, $\psi(t)$ explicitly, the derivatives $\varphi'(t)$, $\psi'(t)$ at point $t = t_0$ can be seen from ODEs.

$$\begin{cases} \varphi'(t) = F(\varphi(t_0), \psi(t_0)) = F(x_0, y_0) \\ \psi'(t) = G(\varphi(t_0), \psi(t_0)) = G(x_0, y_0) \end{cases}$$

→ i.e., the tangent vector of the curve at (x_0, y_0) can be seen. Moreover, note that this tangent vector does not depend on choice of t_0. This yields the following observation:

For any solution of the system, if it's curve over the xy-plane passes through (x_0, y_0) ⟶

→ then the tangent vector of the curve at (x_0, y_0) is $\begin{bmatrix} F(x_0,y_0) \\ G(x_0,y_0) \end{bmatrix}$



* **Direction Field:** the vector field are xy-plane, or, "phase plain", specified by

$$(x_0, y_0) \longrightarrow \begin{bmatrix} F(x_0,y_0) \\ G(x_0,y_0) \end{bmatrix}$$

↑This can ALWAYS be drawn!

* **Integral Curve:** (Trajectory / Orbit) the curve given by one solution to the system.

* **Phase Portrait:** Phase plane + representation set of different integral curve.

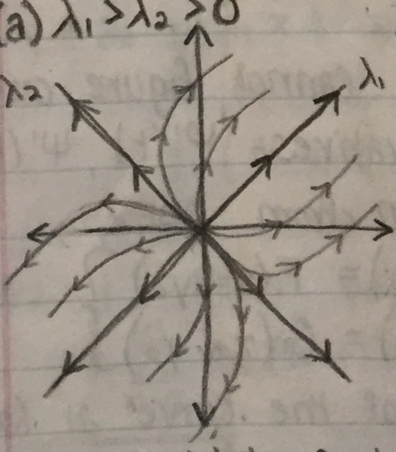
$\boxed{\text{Linear Case}}$

$$\left\{ \begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned} \right\}$$

⇒ λ_1, λ_2 denote the eigenvalues of $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

① **Case #1:** Real, Distinct Eigenvalues

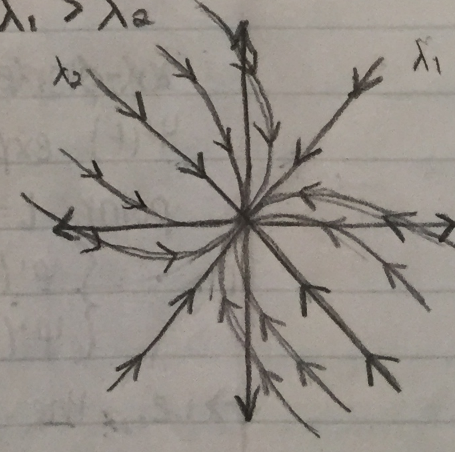
(a) $\lambda_1 > \lambda_2 > 0$



$\vec{x}(t) = C_1e^{\lambda_1 t}\vec{u} + C_2e^{\lambda_2 t}$

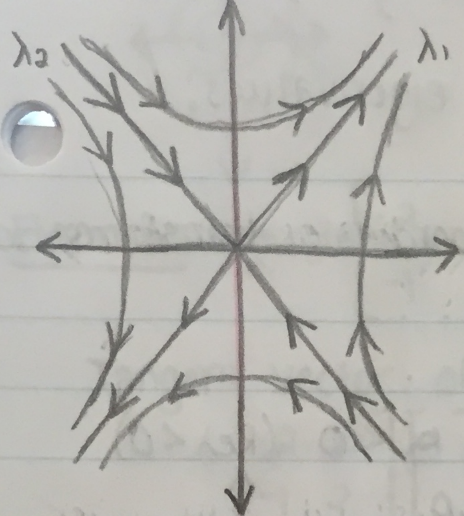
Equilibrium = Nodal Source
⇒ Unstable

(b) $0 > \lambda_1 > \lambda_2$



Equilibrium = Nodal Sink
⇒ Stable
(asymptotically stable)

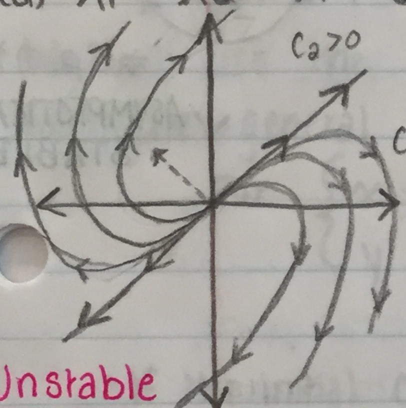
(c) $\lambda_1 > 0 > \lambda_2$



Equilibrium: Saddle Point
$\Rightarrow$ Unstable

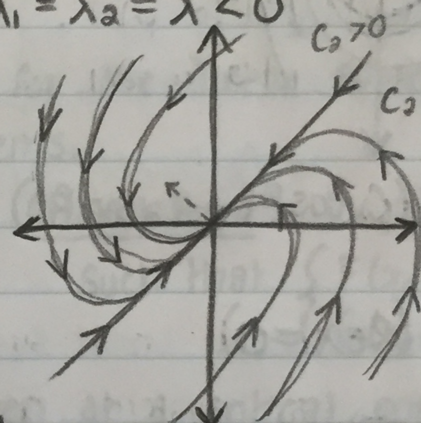
② Case #2: Real, Repeated Eigenvalues

(a) $\lambda_1 = \lambda_2 = \lambda > 0$



Unstable

(b) $\lambda_1 = \lambda_2 = \lambda < 0$



Asymptotically Stable

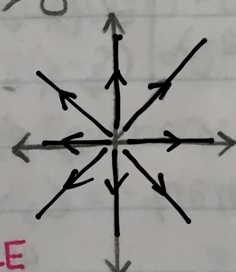
$\boxed{\text{Improper Node}}$

$$\vec{x}(t) = C_1 e^{\lambda t}\vec{v}t + C_2 e^{\lambda t}(t\vec{v} + \vec{w})$$
$$C_2 e^{\lambda t} t \vec{v}$$

(c) $\lambda_1 = \lambda_2 = \lambda$ $\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}$

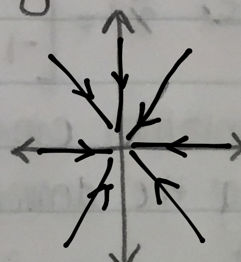
↳ with 2 independent eigenvectors

(i) $\lambda > 0$



UNSTABLE

(ii) $\lambda < 0$



ASYMPTOTIC-ALLY STABLE

$\boxed{\text{Proper Nodes}}$

$$\vec{x}(t) = C_1 e^{\lambda t}\vec{v}_1 t + C_2 e^{\lambda t}\vec{v}_2 t$$

⟶

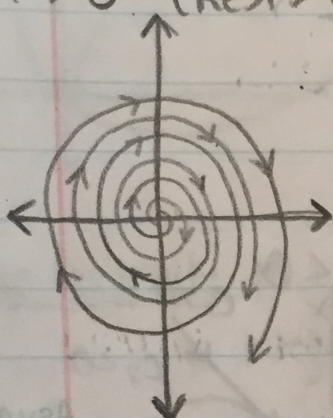
→ Degenerate cases (some eigenvalues being 0) is not used in nonlinear systems

↳ We always consider non-zero eigenvalues!

③ <u>Case #3</u>: Complex Eigenvalue Cases

By an argument involving polar coordinate transformation phase portraits look like...

(a) $\alpha > 0$ ($Re\lambda > 0$)

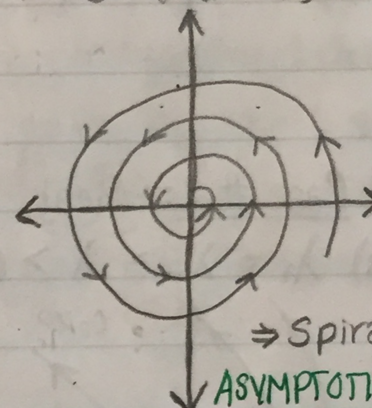


⇒ Spiral Source

UNSTABLE

$\vec{x}(t) = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$

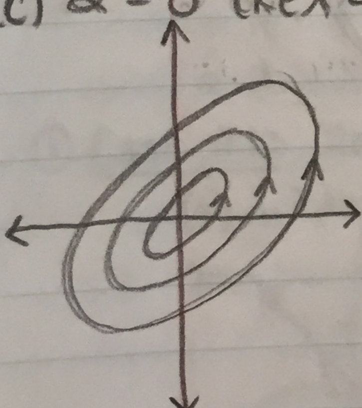
(b) $\alpha < 0$ ($Re\lambda < 0$)



⇒ Spiral Sink

ASYMPTOTICALLY STABLE

(c) $\alpha = 0$ ($Re\lambda = 0$)



⇒ Center Point

STABLE

<u>Example</u>: $\vec{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\vec{x} \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = (\lambda-2)^2 + 1 = 0$

$\Rightarrow \lambda = 2 \pm i$

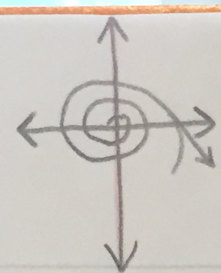
$\Rightarrow Re\lambda = 2 > 0$

↳ Therefore, we will get Spiral Source

** Orientation can be obtained by testing the derivative at one arbitrary point.

At point (1,0), tangent vector

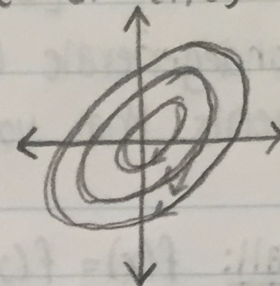
$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$



Example: $\vec{x}' = \begin{bmatrix} -2 & 13 \\ -1 & 2 \end{bmatrix} \vec{x} \Rightarrow \lambda = \pm 3i \Rightarrow \text{Re}\lambda = 0$
↳ Center Point

Tangent vector of integral curve at $(1,0)$

$$\begin{bmatrix} -2 & 13 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$



- We will use the phase portraits for the linear systems to investigate general nonlinear systems.

For $\begin{Bmatrix} x' = F(x,y) \\ y' = G(x,y) \end{Bmatrix}$ Critical Point: (x_0, y_0) such that $\begin{Bmatrix} F(x_0,y_0) = 0 \\ G(x_0,y_0) = 0 \end{Bmatrix}$

- If the initial condition is chosen at a critical point, then the solution is constantly $\begin{Bmatrix} x = x_0 \\ y = y_0 \end{Bmatrix}$

Example: $\begin{Bmatrix} x' = (x+2)(y-1) \\ y' = x(y-2) \end{Bmatrix}$

$\Rightarrow$ To look for critical points, set

$$\begin{Bmatrix} (x+2)(y-1) = 0 \\ x(y-2) = 0 \end{Bmatrix}$$

$\Rightarrow$ 1st Equation: $x = -2$ or $y = 1$

If $x = -2$, the 2nd equation...

$\Rightarrow -2(y-2) = 0$

$\Rightarrow y = 2$

If $y = 1$, the 2nd equation...

$\Rightarrow x(-3) = 0 \Rightarrow x = 0$

→

⇒ Critical Points: $(-2,2)$, $(0,1)$

• **Locally Linear System:**

At each critical point (x_0, y_0), the Jacobian matrix....

$$J(x_0, y_0) = \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix}$$

.... is nondegenerate (meaning $\det \neq 0$), plus other requirements that you will not understand until later classes.

linear organization

* Recall: $f(x) = f(x_0) + f'(x_0)(x - x_0) + O((x-x_0)^2)$

$$\begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix} = \begin{bmatrix} F(x_0,y_0) \\ G(x_0,y_0) \end{bmatrix} + \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Multivariable generalization of linear approx:

$$+ O\left[(x-x_0)^2 + (y-y_0)^2\right]$$

⇒ For locally linear systems near critical point, we use the linear system $\vec{x}' = J(x_0, y_0)\vec{x}$ to approx. the solution